

q-deformed statistical-mechanical structure in the dynamics of the Feigenbaum attractor

A. Robledo

*Instituto de Física,
Universidad Nacional Autónoma de México,
Apartado Postal 20-364, México 01000 D.F., Mexico.**
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We show that the two complementary parts of the dynamics associated to the Feigenbaum attractor, inside and towards the attractor, form together a q -deformed statistical-mechanical structure. A time-dependent partition function produced by summing distances between neighboring positions of the attractor leads to a q -entropy that measures the fraction of ensemble trajectories still away at a given time from the attractor (and the repeller). The values of the q -indexes are given by the attractor's universal constants, while the thermodynamic framework is closely related to that first developed for multifractals.

The Feigenbaum attractor, an icon of the historical developments in the theory of nonlinear dynamics [1], is at present getting renewed attention [2]. This is because it offers a convenient model system to explore features that might reflect those of statistical mechanical systems under conditions of phase space mixing and ergodicity breakdown. It therefore offers insights on the limits of validity of ordinary statistical mechanics. Very recently a thorough description with newly revealed features has been given [3] [4] of the intricate dynamics that takes place both inside and towards this famous multifractal attractor. Here we show that these two different types of dynamics are related to each other in a statistical-mechanical fashion, i.e. the dynamics *at* the attractor provides the 'microscopic configurations' in a partition function while the *approach* to the attractor is economically described by an entropy obtained from it. As we show below, this statistical-mechanical property conforms to a q -deformation [5] [6] of the ordinary exponential (Boltzmann) weight statistics.

Trajectories within a (one-dimensional) multifractal attractor with vanishing Lyapunov exponent λ , such as Feigenbaum's, show self-similar temporal structures, they preserve memory of their previous locations and do not have the mixing property of chaotic trajectories [7]. The fluctuating sensitivity to initial conditions has the form of infinitely many interlaced q -exponential functions that fold into a single one with use of a two-time scaling property [3] [6] [8]. More precisely, there is a hierarchy of such families of interlaced q -exponentials; an intricate (and previously unknown) state of affairs that befits the rich scaling features of a multifractal attractor. Furthermore, the entire dynamics is made of a family of pairs of Mori's dynamical q -phase transitions [7] [3] [6] [8].

On the other hand, the process of convergence of trajectories into the Feigenbaum attractor is governed by another unlimited hierarchy feature built into the preimage structure of the attractor and its counterpart repeller [4]. The overall rate of approach of trajectories towards the attractor (and to the repeller) is conveniently measured

by the fraction of (fine partition) bins W_{t_1} still occupied at time t_1 by an ensemble of trajectories with initial positions uniformly distributed over the entire phase space [9] [4]. For the first few time steps the rate W_{t_1} remains constant, $W_{t_1} \simeq \Delta$, $1 \leq t_1 \leq t_0$, $t_0 = O(1)$ [10], after which a power-law decay with log-periodic modulation sets in, a signature of discrete-scale invariance [11]. This property of W_{t_1} is explained in terms of a sequential formation of gaps in phase space, and its self-similar features are seen to originate in the mentioned ladder feature of the preimage structure [4]. The rate W_{t_1} was originally presented in Ref. [9] where the power law exponent φ in

$$W_{t_1} \simeq \Delta h\left(\frac{\ln t}{\ln \Lambda}\right) t^{-\varphi}, \quad t = t_1 - t_0, \quad (1)$$

was estimated numerically. Above, $h(x)$ is a periodic function with $h(1) = 1$, and Λ is the scaling factor between the periods of two consecutive oscillations [12].

We proceed now to demonstrate the connection between the aforementioned dynamical properties. We recall [1] the definition of the interval lengths or diameters $d_{n,m}$ that measure the bifurcation forks that form the period-doubling cascade sequence in unimodal maps, here represented by the logistic map $f_\mu(x) = 1 - \mu x^2$, $-1 \leq x \leq 1$, $0 \leq \mu \leq 2$. These quantities are measured when considering the superstable periodic orbits of lengths 2^n , $n = 1, 2, 3, \dots$; i.e., the 2^n -cycles that contain the point $x = 0$ at $\mu_n < \mu_\infty$, where $\mu_\infty = 1.401155189\dots$ is the value of the control parameter μ at the period-doubling accumulation point [13]. The positions of the limit 2^∞ -cycle constitute the Feigenbaum attractor. The $d_{n,m}$ in these orbits are defined (here) as the (positive) distances of the elements x_m , $m = 0, 1, 2, \dots, 2^{n-1} - 1$, to their nearest neighbors $f_{\mu_n}^{(2^{n-1})}(x_m)$, i.e.,

$$d_{n,m} \equiv \left| f_{\mu_n}^{(m+2^{n-1})}(0) - f_{\mu_n}^{(m)}(0) \right|. \quad (2)$$

For large n , $d_{n,0}/d_{n+1,0} \simeq \alpha$, where α is Feigenbaum's universal constant $\alpha \simeq 2.5091$.

Innermost to our arguments is the following comprehensive property: Time evolution at μ_∞ from $t = 0$ up to $t \rightarrow \infty$ traces the period-doubling cascade progression from $\mu = 0$ up to μ_∞ . Not only is there a close resemblance between the two developments but also asymptotic quantitative agreement. For example, the trajectory inside the Feigenbaum attractor with initial condition $x_0 = 0$, the 2^∞ -supercycle orbit, takes positions x_t such that the distances between nearest neighbor pairs of them reproduce the diameters $d_{n,m}$ defined from the supercycle orbits with $\mu_n < \mu_\infty$. See Fig. 1, where the absolute value of positions and logarithmic scales are used to illustrate the equivalence. This property has been central to obtain rigorous results for the fluctuating sensitivity to initial conditions $\xi_t(x_0)$ within the Feigenbaum attractor, as separations at chosen times t of pairs of trajectories originating close to x_0 can be obtained as diameters $d_{n,m}$, where n and m relate to t and x_0 , respectively [3] [6].

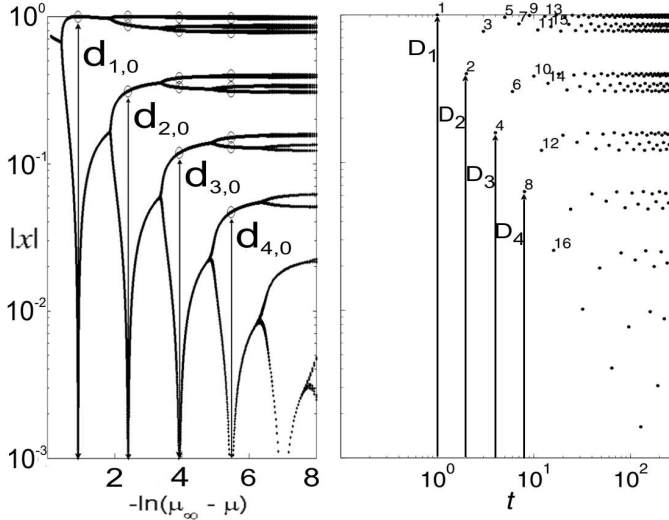


FIG. 1: Left panel: Absolute value of attractor positions for the logistic map $f_\mu(x)$ in logarithmic scale as a function of $-\ln(\mu_\infty - \mu)$. Right panel: Absolute value of trajectory positions for $f_\mu(x)$ at μ_∞ with initial condition $x_0 = 0$ in logarithmic scale as a function of the logarithm of time t , also shown by the numbers close to the circles. The arrows indicate the equivalence between the diameters $d_{n,0}$ in the left panel, and position differences D_n with respect to $x_0 = 0$ in the right panel.

Further, the complex dynamical events that fix the decay rate W_{t_1} can be understood in terms of the correlation between time evolution at μ_∞ from $t = 0$ up to $t \rightarrow \infty$ and the ‘static’ period-doubling cascade progression from $\mu = 0$ up to μ_∞ . As shown recently [14] each doubling of the period (obtained through the shift $\mu_n \rightarrow \mu_{n+1}$) introduces additional elements in the hierarchy of the preimage structure and in the family of sequentially-formed phase space gaps in the finite period

cycles. The complexity of these added elements is similar to that of the total period 2^n system. Also, the shift $\mu_n \rightarrow \mu_{n+1}$ increases in one unit the number of undulations in the transitory log-periodic power law decay found for the corresponding rate W_{n,t_1} of approach to the 2^n -supercycle attractor [14]. As a consequence of this we have obtained detailed understanding of the mechanism by means of which the discrete scale invariance implied by the log-periodic property in $W_{t_1} \equiv \lim_{n \rightarrow \infty} W_{n,t_1}$ arises. What is more, the rate W_{t_1} , at the values of time for period doubling, can be obtained quantitatively from the supercycle diameters $d_{n,m}$, that is [4], $W_{t_1} = \Delta Z_t$, $t = t_1 - t_0$, and

$$Z_t = \sum_{m=0}^{2^{n-1}-1} d_{n,m}, \quad t = 2^{n-1}, \quad n = 1, 2, 3, \dots \quad (3)$$

Eq. (3) is an explicit expression equivalent to the numerical procedure followed in Ref. [15] by the use of the triadic cantor set construction of the Feigenbaum attractor to evaluate the power law exponent φ , and from which the value for $\varphi \cong 0.800138194$ is reported.

Now, to reveal the aforesaid statistical-mechanical structure we identify the decay rate Z_t as a partition function. From this perspective the diameters $d_{n,m}$ are configurational terms and we go forward to determine their time dependence through their indexes n and m . The $d_{n,m}$ scale with n for m fixed as

$$d_{n,m} \simeq \alpha_m^{-n+1}, \quad n = 1, 2, 3, \dots, \quad (4)$$

the α_m are universal constants obtained, for instance, from the finite jump discontinuities of Feigenbaum’s trajectory scaling function $\sigma(y) = \lim_{n \rightarrow \infty} d_{n,m+1}/d_{n,m}$, $y = \lim_{n \rightarrow \infty} m/2^n$ [1]. The largest discontinuities of $\sigma(y)$ correspond to the most sparse and the most crowded regions of the multifractal attractor, and for these we have, respectively, $d_{n,0} \simeq \alpha^{-n+1}$ and $d_{n,1} \simeq \alpha^{-2(n-1)}$. (The 1st diameter $d_{0,0} = 1$ and the equality in Eq. (4) is rapidly reached as n increases). With use of the identity $A^{-n+1} \equiv (1 + \beta)^{-\ln A / \ln 2}$, $\beta = 2^{n-1} - 1$, the power law in Eq. (4) can be rewritten as a q -exponential ($\exp_q(x) \equiv [1 - (q-1)x]^{-1/(q-1)}$), i.e.,

$$d_{n,m} \simeq \exp_{q_m}(-\beta \nu_m), \quad (5)$$

where $q_m = 1 + \nu_m^{-1}$, $\nu_m = \ln \alpha_m / \ln 2$, and $\beta = t - 1 = 2^{n-1} - 1$. Likewise, the partition function $Z_t \simeq t^{-\varphi}$ (or $Z_t \simeq \varepsilon^{-n+1}$), with $\varphi = \ln \varepsilon / \ln 2$ and $t = 2^{n-1}$, can be written as

$$Z_t \simeq \exp_Q(-\beta \varphi), \quad (6)$$

$Q = 1 + \varphi^{-1}$ and again $\beta = t - 1 = 2^{n-1} - 1$.

Our main point in this Letter becomes apparent when Eqs. (5) and (6) are used in Eq. (3), to yield

$$\exp_Q(-\beta \varphi) \simeq \sum_m \exp_{q_m}(-\beta \nu_m). \quad (7)$$

Eq. (7) resembles a basic statistical-mechanical expression where the quantities in it play the following roles: β an inverse temperature, φ a free energy (or the product $s = -\beta\varphi$ a Massieu thermodynamic potential, or entropy), and the ν_m configurational energies. However, the equality involves q -deformed exponentials in place of ordinary exponential functions that would be recovered when $Q = q_m = 1$. It is worth noticing that there is a multiplicity of q -indexes associated to the configurational weights in Eq. (7), however their values form a well-defined family [6] determined by the discontinuities of Feigenbaum's function σ . To substantiate the usefulness and appropriateness of this identification we present in the remaining part of this Letter: 1) A 'mean field' evaluation of Z_t and a thermodynamic interpretation of the time evolution process. 2) The relationship of Z_t with the familiar partition function developed for the description of multifractal geometry. 3) A crossover to $q = 1$ ordinary statistics. And 4) a commentary on the implications of our results for the q -deformed generalized statistical mechanical formalism [5].

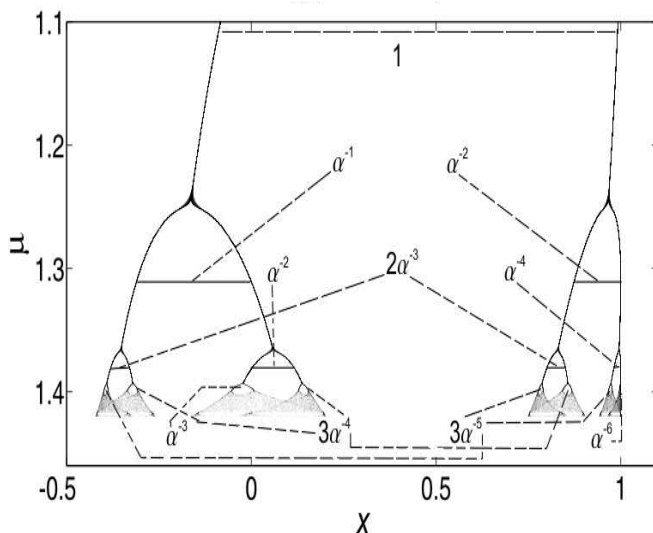


FIG. 2: Sector of the bifurcation tree for the logistic map $f_\mu(x)$ that shows the formation of a Pascal triangle of diameter lengths according to the scaling approximation explained in the text, $\alpha \simeq 2.5091$ is the pertinent universal constant.

Akin to a mean field approximation we assume that, for a given value of n , e.g., $n = 3$, the diameters $d_{n,m}$ that are of comparable lengths have equal length and this is obtained from those of the shortest or longest diameters via a simple scale factor; e.g., $d_{3,3} = d_{3,2} = \alpha^{-1}d_{3,0} = \alpha d_{3,1}$. This introduces some degeneracy in the lengths that propagates across the bifurcation tree. See Fig. 2 [16]. Specifically, the $d_{n,m}$ scale now with increasing n according to a binomial combination of the scaling of those diameters that converge to the most crowded and most sparse regions of the multifractal attractor. This is to

consider that the 2^{n-1} diameters at the n -th supercycle have lengths equal to $\alpha^{-(n-1-l)}\alpha^{-2l}$ and occur with multiplicities $\binom{n-1}{l}$, where $l = 0, 1, \dots, n-1$. As seen in Fig. 2 the diameters form a Pascal triangle across the bifurcation cascade. The partition function can be immediately evaluated to yield

$$Z_t = \sum_{l=0}^{n-1} \binom{n-1}{l} \alpha^{-(n-1-l)} \alpha^{-2l} = (\alpha^{-1} + \alpha^{-2})^{n-1}, \quad (8)$$

$t = 2^{n-1}$. We obtain $\varepsilon = (\alpha^{-2} + \alpha^{-1}) = 1.7883$, $\varphi = 0.8386$, and $Q = 2.1924$, a surprisingly good approximation when compared to the numerical estimates $\varphi = 0.8001$, and $Q = 2.2498$ of the exact values. Under this approximation all the indexes q_m in Eq. (7) are equal, $q_m = q = 1 + \nu^{-1}$, $\nu = \ln \alpha / \ln 2$, and Eq. (7) becomes

$$\exp_Q(-\beta\varphi) = \sum_{l=0}^{n-1} \Omega(n-1, l) \exp_q(-\beta\nu), \quad (9)$$

where $\Omega(n-1, l) = \alpha^{-l} \binom{n-1}{l}$. Thermodynamically, the approach to the attractor described by Eqs. (7) or (9) is a 'cooling' process $\beta \rightarrow \infty$ in which the free energy (or energy) φ is fixed and therefore the entropy $s = -\beta\varphi$ is linear in β . It is illustrative to define an 'energy landscape' for the Feigenbaum attractor as being composed by an infinite number of 'wells' whose equal-valued minima at $\beta \rightarrow \infty$ coincide with the points of the attractor on the interval $[-\alpha^{-1}, 1]$ [12]. When $\beta = 2^{n-1} - 1$, n finite, the wells merge into 2^{n-1} intervals of widths equal to the diameters $d_{n,m}$.

As it is well-known the geometric properties of multifractals conform to a statistical mechanical framework, the so-called thermodynamic formalism [13]. The partition function devised to study their properties, such as the spectrum of singularities $f(\tilde{\alpha})$ [13], is

$$Z(\tau, q) \equiv \sum_m^M p_m^\tau l_m^{-q}, \quad (10)$$

where the l_m (in one-dimensional systems) are M disjoint interval lengths that cover the multifractal set and the p_m are probabilities assigned to these intervals. The usual procedure consists of requiring that $Z(\tau, q)$ neither vanishes nor diverges in the limit $l_m \rightarrow 0$ for all m (and consequently $M \rightarrow \infty$). In this case the exponents τ and q define a function $\tau(q)$ from which $f(\tilde{\alpha})$ is obtained via Legendre transformation [13]. When the multifractal is an attractor its elements become ordered dynamically, and for the Feigenbaum attractor the trajectory with initial condition $x_0 = 0$ generates sequentially the positions that form the diameters, producing all diameters $d_{n,m}$ for n fixed between times $t = 2^{n-1}$ and $t = 3 \cdot 2^{n-1}$. Since the diameters cover the attractor

it is therefore natural to choose the covering lengths at stage n to be $l_m^{(n)} = d_{n,m}$ and to assign to each of them the same probability $p_m^{(n)} = 1/2$. For instance, within the two-scale approximation to the Feigenbaum multifractal [13], $l_k^{(n)} = \alpha^{-(n-1-k)}\alpha^{-2k}$, the condition $Z(\tau, \mathbf{q}) = 1$ reproduces Eq. (8) when $p_m^{(n)} = t^{-1} = 2^{-n+1}$, with $\tau = 1$ and $\mathbf{q} = -\varphi$. It should be kept in mind that the ‘static’ partition function $Z(\tau, \mathbf{q})$ is not meant to distinguish between chaotic and critical (vanishing λ) multifractal attractors as we do here. As we emphasize below, it is the functional form of the link between the probabilities $p_m^{(n)}$ and actual time t that determines the nature of the statistical mechanical structure of the dynamical system.

The recursive method of backward iteration of chaotic maps provides a convenient way for reconstructing multifractal sets and obtaining their underlying statistical mechanics [17]. A chaotic unimodal map has a two-valued inverse and given a position $x = x_n$ a binary tree is formed under backward iteration, so there are 2^n initial conditions x_0 for trajectories that lead to x_n . Since the Lyapunov exponent is positive $\lambda > 0$, for large n lengths expand under forward iteration according to $l \sim \exp(\lambda n)$ and contract under backward iteration as $l \sim \exp(-\lambda n)$. We can define, as above, a set of covering lengths $\delta_{n,m} = \exp(-\lambda_m n)$, where λ_m is a local Lyapunov exponent with $\lambda_m \rightarrow \lambda$ as $n \rightarrow \infty$ and where m relates to the initial condition x_0 . Use of these lengths in a partition function like that in Eq. (3) gives

$$\exp(-\beta\Phi) = \sum_m \exp(-\beta\lambda_m), \quad (11)$$

where now $\beta = n$. Recalling Pesin’s theorem [13], Φ is clearly related, for large n , to the Kolmogorov-Sinai entropy. The crossover from q -deformed statistics to ordinary $q = 1$ statistics can be observed for control parameter values in the vicinity of the Feigenbaum attractor, $\mu \gtrsim \mu_\infty$, when the attractor consists of 2^π bands. The Lyapunov coefficient λ of the chaotic attractor decreases with $\Delta\mu = \mu - \mu_\infty$ as $\lambda \propto 2^{-\pi} \sim \Delta\mu^\kappa$, $\kappa = \ln 2 / \ln \delta_F(\zeta)$, where δ_F is the Feigenbaum constant that measures the rate of development of the bifurcation tree in control parameter space [1]. The chaotic orbit consists of an interband periodic motion of period 2^π and an intraband chaotic motion. The expansion rate $\sum_{i=0}^{t-1} \ln |df_\mu(x_i)/dx_i|$ fluctuates with increasing amplitude as $\ln t$ for $t < 2^\pi$ but converges to a fixed number that grows linearly with t for $t \gg 2^\pi$ [7]. This translates as dynamics with $q \neq 1$ for $t < 2^\pi$ but ordinary dynamics with $q = 1$ for $t \gg 2^\pi$ [6].

We have shown that there is a statistical-mechanical property lying beneath the dynamics of an ensemble of trajectories en route to the Feigenbaum attractor (and repeller). The fraction of phase space still occu-

pied at time t is a partition function Z_t made up of q -exponential weighted configurations, while Z_t itself is the q -exponential of a thermodynamic potential function. This is a clear signature of q -deformation of ordinary statistical mechanics, and, to our knowledge, it is the first *bona fide* concrete instance (anticipated or not in the form presented here) where arguments can be made explicit and rigorous. There is a close parallel with the thermodynamic formalism for multifractal sets, but it should be emphasized that the departure from the usual exponential statistics is dynamical in origin, and due to the vanishing of the (only) Lyapunov exponent. Our results suggest a novel variant for the theory of large deviations, the mathematical articulation of statistical mechanics [18].

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* Electronic address: robledo@fisica.unam.mx

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